

Dirichlet Problem for Coupled Elliptic Systems

Azeddine Baalal* and Mohamed Berghout†

Department of Mathematics-Laboratory MACS

Faculty of Sciences Aïn Chock, University Hassan II of Casablanca

B.P 5366 Maarif Casablanca 20100 - Morocco

2016

Abstract

Let Ω be a bounded domain in \mathbb{R}^d ($d \geq 2$) pretty regular. In this paper we solve the variational Dirichlet problem for a class of quasilinear elliptic systems.

Introduction

In the classical theory of the Laplace equation several main parts of mathematics are joined in a fruitful way: Calculus of Variations, Partial Differential Equations, Potential Theory, Function Theory (Analytic Functions), Mathematical Physics and Calculus of Probability. This is the strength of the classical theory. The p -Laplace equation occupies a similar position, when it comes to non-linear phenomena.

Problem involving the p -Laplacian operator appears in pure mathematics such as the theory of quasiregular and quasiconformal mapping as well as in applied mathematics. Indeed, it intervenes in numerous fields in experimental sciences: nonlinear reaction-diffusion problems, dynamics of populations, non-Newtonian fluids, ect....for this reasons the word “ p -Laplacian” has become a key word in nonlinear analysis and problems involving this second order quasilinear operator are now extensively studied in the literature. This elliptic operator it generalizes the usual Laplace operator $\Delta = \Delta_2$

*a.baalal@fsac.ac.ma

†moh.berghout@gmail.com

whose study has been widely discussed in recent decades but the lack of the Hilbert structure of the space $W_0^{1,p}(\Omega)$ when passing from $p = 2$ to $p \neq 2$ make his study very difficult.

The notion of the solution of quasilinear homogeneous Dirichlet problem associated with the p -poisson equation

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega ; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

is always understood in a weak sense, precisely is a function $u \in W_0^{1,p}(\Omega)$.

In literature, there exists numerous papers dedicated to the study of equations and systems involving the p -laplacian operator. In fact the study of scalar equations had really started in the middle of 80s by M. Ôtani [23] in one dimension then in dimension n by F. de Thélin [10] who obtained the first results on the equation of the form: $-\Delta_p u = \lambda u^{\gamma-1}$, after the existence and the uniqueness of radial solutions in \mathbb{R}^n have showed independently by the last author and W. M. Ni. Serrin [25]. this result has been generalized by M. Ôtani [24] to any arbitrary open subset of \mathbb{R}^n . In 1987, F. de Thélin [11] has extended these results to the equation of the type $\Delta_p u = g(x, u)$ where g is a function controlled by polynomial functions in u . In addition, there are other results on the uniqueness were stated by J. I. Díaz and J. E. Saa [14] in 1987 for the equation $-\Delta_p u = f(x, u)$.

The case of systems presents a new challenge and leads to tremendous complications related to the coupling and lot of work done in this area for instance we cite [12], [16], [13], [7], [5], [17] and [4].

Let Ω be a bounded domain in \mathbb{R}^d ($d \geq 2$) pretty regular, and let $\mathcal{L}_1, \mathcal{L}_2$ are a quasi-linear elliptic differential operators in divergence form

$$\begin{cases} \mathcal{L}_1(u, v) := -\Delta_p u + \varphi(., u, v) \\ \mathcal{L}_2(u, v) := -\Delta_p v + \psi(., u, v) \end{cases}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ denotes the p -Laplacian operator and $\varphi, \psi : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given Carathéodory functions satisfying:

$$\begin{aligned} (H_1) \quad & |\varphi(x, u, v)| \leq a_1 |u|^{p-1} + a_2 |v|^{p-1} ; \\ & |\psi(x, u, v)| \leq b_1 |v|^{p-1} + b_2 |u|^{p-1} . \end{aligned}$$

where a_1, a_2, b_1, b_2 are positive constants.

(H_2) $s \mapsto \varphi(x, s, t)$ and $s \mapsto \psi(x, s, t)$ are increasing for all $t \in \mathbb{R}$;
 $t \mapsto \varphi(x, s, t)$ and $t \mapsto \psi(x, s, t)$ are increasing for all $s \in \mathbb{R}$.

Our aim in this paper is to solve the Dirichlet problem

$$\begin{cases} \mathcal{L}_1(u, v) := -\Delta_p u + \varphi(., u, v) = 0, & \text{in } \Omega ; \\ \mathcal{L}_2(u, v) := -\Delta_p v + \psi(., u, v) = 0, & \text{in } \Omega ; \\ u = h, v = k, & \text{on } \partial\Omega. \end{cases} \quad (1)$$

with a continuous data boundary.

This paper consists of three sections. First, we give some definitions for the weak subsolutions, supersolution and solution of the system (1) and we prove that if a pair of functions (u, v) is a supersolution (resp. subsolution) of (1), then the pair of functions $(u + \alpha, v + \beta)$ (resp. $(u - \alpha, v - \beta)$) is also a supersolution (resp. subsolution) of (1) for every $\alpha > 0$ and $\beta \geq 0$. In section two we show that the comparison principle holds for \mathcal{L}_1 and \mathcal{L}_2 . After this preparation we are able in section three to solve the Dirichlet problem. So at first by the Shaulder's fixed point theorem we prove the existence of solutions to the associated variational problem for Ω with $|\Omega|$ is small enough, after we solve the Dirichlet problem with continuous data boundary for Ω p -regular with $|\Omega|$ is small enough. Finally in the general case we will approach Ω by an compact K such that $\Omega \setminus K$ be p -regular with $|\Omega|$ is small enough. Then by an argument of collection of K with a p -regular set who have a rather small Lebesgue measure and by choicing a sequence of functions and using a diagonal extraction process; we show that the limit of a sequence of functions it's the solution.

Notation

Throughout of this paper we will use the following notation: \mathbb{R}^d is the real Euclidean d -space, $d \geq 2$. Ω is a open bounded domaine of \mathbb{R}^d pretty regular. For every $p \in [1; +\infty[$ we denote by $p' = \frac{p}{p-1}$ the Hölder conjugate of p . For a measurable set X and for $p \geq 1$, $L^p(X)$ is the p^{th} -power Lebesgue space defined on X and $L^{p'}(X)$ denotes the dual space of $L^p(X)$. For an open set U of \mathbb{R}^d , we denote by $\mathcal{C}^k(U)$ the set of functions which k -th derivative is continuous for k positive integer, $\mathcal{C}^\infty(U) = \cap_{k \geq 1} \mathcal{C}^k(U)$ and by $\mathcal{C}_c^\infty(U)$ the set of all functions in $\mathcal{C}^\infty(U)$ with compact support. The Sobolev space $\mathcal{W}^{1,p}(U)$ is the Banach space of all functions $u \in L^p(U)$ whose gradient in the distribution sense $\nabla u \in (L^p(U))^d$, equipped with the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$, $\mathcal{W}_{loc}^{1,p}(U)$ is the space of all functions $u \in \mathcal{W}^{1,p}(U)$

for every open $U \subset \overline{U} \subset \Omega$. $\mathcal{W}_0^{1,p}(U)$ is the closure of $\mathcal{C}_c^\infty(U)$ in $\mathcal{W}^{1,p}(U)$ relatively to its norm (for more details see for example [20], [21], [22], [9], [6]). For $E \subset \mathbb{R}^d$ measurable we denote by $|E|$ the Lebesgue measure of E . If $A \subset \mathbb{R}^d$ we denote by \overline{A} the topological closure of A and by ∂A the topological boundary of A . for a function f we denote $f^+ := \max\{f, 0\}$, and we design by \limsup (resp. \liminf) the upper limit (resp. lower limit) of a real function. $\mathcal{C}(X)$ the set of all continuous functions on X . The order on the set of pairs of functions on a set M is the usual order product:

$$(f, g) \leq (h, k) \Leftrightarrow f \leq h \text{ and } g \leq k.$$

1 Solution of (1)

Definition 1 We say that a pair of functions $(u, v) \in \mathcal{W}_{loc}^{1,p}(\Omega) \times \mathcal{W}_{loc}^{1,p}(\Omega)$ is a weak solution of (1) if

$$\begin{cases} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \eta_1 dx + \int_{\Omega} \varphi(x, u, v) \eta_1 dx = 0, \\ \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \eta_2 dx + \int_{\Omega} \psi(x, u, v) \eta_2 dx = 0, \end{cases}$$

for all $\eta_1, \eta_2 \in \mathcal{W}_0^{1,p}(\Omega)$.

We say that a pair of functions $(u, v) \in \mathcal{W}_{loc}^{1,p}(\Omega) \times \mathcal{W}_{loc}^{1,p}(\Omega)$ is a supersolution (resp. subsolution) of (1) if

$$\begin{cases} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \eta_1 dx + \int_{\Omega} \varphi(x, u, v) \eta_1 dx \geq 0 \\ \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \eta_2 dx + \int_{\Omega} \psi(x, u, v) \eta_2 dx \geq 0 \end{cases} \quad (\text{resp. } \begin{cases} \leq 0 \\ \leq 0 \end{cases})$$

for all nonnegative functions $\eta_1, \eta_2 \in \mathcal{W}_0^{1,p}(\Omega)$.

Proposition 2 If a pair of functions (u, v) is a supersolution (resp. subsolution) of (1). then the pair of functions $(u + \alpha, v + \beta)$ (resp. $(u - \alpha, v - \beta)$) is also a supersolution (resp. subsolution) of (1) for every $\alpha > 0$ and $\beta \geq 0$.

Proof. Let $\eta_1, \eta_2 \in \mathcal{W}_0^{1,p}(\Omega)$ two non-negative tests functions, $\alpha > 0$ and $\beta \geq 0$. Then

$$\begin{cases} \int_{\Omega} |\nabla(u + \alpha)|^{p-2} \nabla(u + \alpha) \nabla \eta_1 dx + \int_{\Omega} \varphi(x, u + \alpha, v + \beta) \eta_1 dx = \\ \int_{\Omega} |\nabla(u + \alpha)|^{p-2} \nabla(u + \alpha) \nabla \eta_1 dx + \int_{\Omega} \varphi(x, u, v) \eta_1 dx + \\ \int_{\Omega} (\varphi(x, u + \alpha, v + \beta) - \varphi(x, u, v + \beta)) \eta_1 dx + \int_{\Omega} (\varphi(x, u, v + \beta) - \varphi(x, u, v)) \eta_1 dx \end{cases}$$

and

$$\left\{ \begin{array}{l} \int_{\Omega} |\nabla(v + \beta)|^{p-2} \nabla(v + \beta) \nabla \eta_2 dx + \int_{\Omega} \psi(x, u + \alpha, v + \beta) \eta_2 dx = \\ \int_{\Omega} |\nabla(v + \beta)|^{p-2} \nabla(v + \beta) \nabla \eta_2 dx + \int_{\Omega} \psi(x, u, v) dx + \\ \int_{\Omega} (\psi(x, u + \alpha, v + \beta) - \psi(x, u, v + \beta)) \eta_2 dx + \int_{\Omega} (\psi(x, u, v + \beta) - \psi(x, u, v)) \eta_2 dx \end{array} \right.$$

since $\nabla(u + \alpha) = \nabla u$ and $\nabla(v + \beta) = \nabla v$, then

$$\begin{aligned} & \int_{\Omega} |\nabla(u + \alpha)|^{p-2} \nabla(u + \alpha) \nabla \eta_1 dx + \int_{\Omega} \varphi(x, u + \alpha, v + \beta) \eta_1 dx = \\ & \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \eta_1 dx + \int_{\Omega} \varphi(x, u, v) \eta_1 dx + \int_{\Omega} (\varphi(x, u + \alpha, v + \beta) - \varphi(x, u, v + \beta)) \eta_1 dx \\ & + \int_{\Omega} (\varphi(x, u, v + \beta) - \varphi(x, u, v)) \eta_1 dx \end{aligned}$$

and

$$\left\{ \begin{array}{l} \int_{\Omega} |\nabla(v + \beta)|^{p-2} \nabla(v + \beta) \nabla \eta_2 dx + \int_{\Omega} \psi(x, u + \alpha, v + \beta) \eta_2 dx = \\ \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \eta_2 dx + \int_{\Omega} \psi(x, u, v) dx + \\ \int_{\Omega} (\psi(x, u + \alpha, v + \beta) - \psi(x, u, v + \beta)) \eta_2 dx + \int_{\Omega} (\psi(x, u, v + \beta) - \psi(x, u, v)) \eta_2 dx \end{array} \right.$$

using (H_2) and the fact that (u, v) is a supersolution, we get

$$\left\{ \begin{array}{l} \int_{\Omega} |\nabla(u + \alpha)|^{p-2} \nabla(u + \alpha) \nabla \eta_1 dx + \int_{\Omega} \varphi(x, u + \alpha, v + \beta) \eta_1 dx \geq 0 \\ \int_{\Omega} |\nabla(v + \beta)|^{p-2} \nabla(v + \beta) \nabla \eta_2 dx + \int_{\Omega} \psi(x, u + \alpha, v + \beta) \eta_2 dx \geq 0 \end{array} \right.$$

which shows that $(u + \alpha, v + \beta)$ is a supersolution. By the same way we show that $(u - \alpha, v - \beta)$ is a subsolution. \square

2 Variational Dirichlet problem

We put:

$$s := \frac{dr(p-1)}{d-pr} \text{ where } r \in \left[\frac{dp'}{d+p'}; \frac{d}{p} \right], 1 < p < d \text{ and } \frac{d}{p} \leq p' \text{ for some } d.$$

Let $f, g \in L^r(\Omega) \cap L^{p'}(\Omega)$ and $h, k \in \mathcal{W}^{1,p}(\Omega)$. For $f \in L^r(\Omega)$ let $\tilde{u}_f \in \mathcal{W}_o^{1,p}(\Omega)$ the solution of

$$\left\{ \begin{array}{l} \Delta_p u = f \text{ in } \Omega, \\ u - h \in \mathcal{W}_o^{1,p}(\Omega). \end{array} \right.$$

and for $g \in L^r(\Omega)$ let $\tilde{v}_g \in \mathcal{W}^{1,p}(\Omega)$ the solution of

$$\begin{cases} \Delta_p v = g \text{ in } \Omega, \\ v - k \in \mathcal{W}_0^{1,p}(\Omega). \end{cases}$$

Consider the following Dirichlet problem

$$(P) \begin{cases} \mathcal{L}_1(u, v) := -\Delta_p u + \varphi(x, u, v) = 0 \text{ a.e. } x \in \Omega, \\ \mathcal{L}_2(u, v) := -\Delta_p v + \psi(x, u, v) = 0 \text{ a.e. } x \in \Omega, \\ u - h \in \mathcal{W}_0^{1,p}(\Omega) \text{ and } v - k \in \mathcal{W}_0^{1,p}(\Omega) \text{ on } \partial\Omega. \end{cases}$$

We set $\tilde{u} = u - h$ and $\tilde{v} = v - k$, then we take back to the following homogeneous variational problem:

$$(\tilde{P}) \begin{cases} \mathcal{L}_1(\tilde{u}, \tilde{v})(x) := -\Delta_p(\tilde{u} + h) + \tilde{\varphi}(x, \tilde{u}, \tilde{v}) = 0 \text{ a.e. } x \in \Omega \\ \mathcal{L}_2(\tilde{u}, \tilde{v})(x) := -\Delta_p(\tilde{v} + k) + \tilde{\psi}(x, \tilde{u}, \tilde{v}) = 0 \text{ a.e. } x \in \Omega \\ \tilde{u} \in \mathcal{W}_0^{1,p}(\Omega), \tilde{v} \in \mathcal{W}_0^{1,p}(\Omega) \end{cases}.$$

where

$$\begin{aligned} \tilde{\varphi}(x, u, v) &= \varphi(x, u + h, v + k), \\ \tilde{\psi}(x, u, v) &= \psi(x, u + h, v + k). \end{aligned}$$

Remark 3 • Using (H_1) and inequality

$$|a + b|^p \leq \begin{cases} (1 + \varepsilon)^{p-1} |a|^p + \left(1 + \frac{1}{\varepsilon}\right)^{p-1} |b|^p & \text{for } 1 \leq p < \infty; \\ |a|^p + |b|^p & \text{for } 0 < p < 1. \end{cases}$$

for arbitrary $a, b \in \mathbb{R}$ and $\varepsilon > 0$, we obtain that $\tilde{\varphi}, \tilde{\psi} : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given Carathéodory functions satisfying:

$$(\tilde{H}) \begin{cases} |\tilde{\varphi}(x, u, v)| \leq a'_1 |u|^{p-1} + a'_2 |v|^{p-1} + c(x); \\ |\tilde{\psi}(x, u, v)| \leq b'_1 |u|^{p-1} + b'_2 |v|^{p-1} + c'(x). \end{cases}$$

almost everywhere $x \in \Omega$. Where

$$\begin{aligned} a'_1 &= a_1 (1 + \varepsilon)^{p-1}, \quad a'_2 = a_2 (1 + \varepsilon)^{p-1}, \\ b'_1 &= b_1 (1 + \varepsilon)^{p-1}, \quad b'_2 = b_2 (1 + \varepsilon)^{p-1}, \\ c(x) &= a_1 \left(1 + \frac{1}{\varepsilon}\right)^{p-1} |h(x)|^{p-1} + a_2 \left(1 + \frac{1}{\varepsilon}\right)^{p-1} |k(x)|^{p-1}, \\ c'(x) &= b_1 \left(1 + \frac{1}{\varepsilon}\right)^{p-1} |h(x)|^{p-1} + b_2 \left(1 + \frac{1}{\varepsilon}\right)^{p-1} |k(x)|^{p-1}. \end{aligned}$$

- If (\tilde{u}, \tilde{v}) is a solution of (\tilde{P}) . $(\tilde{u} + h, \tilde{v} + k)$ is a solution of (P) .
- $\tilde{\varphi}(x, u, v), \tilde{\psi}(x, u, v) \in L^{p'}(\Omega)$.

Theorem 4 *If $|\Omega|$ is small enough. Then system (P) admits a weak solution.*

Proof

We consider the operator:

$$\begin{aligned} T : L^r(\Omega) \times L^r(\Omega) &\rightarrow L^p(\Omega) \times L^p(\Omega) \\ (f, g) &\mapsto (\tilde{u}_f, \tilde{v}_g) \end{aligned}$$

Lemma 5 *The operator $T : L^r(\Omega) \times L^r(\Omega) \rightarrow L^p(\Omega) \times L^p(\Omega)$ is completely continuous.*

Proof. By the regularity theory we have \tilde{u}_f and \tilde{v}_g are continuous functions, so $T : L^r(\Omega) \times L^r(\Omega) \rightarrow \mathcal{W}_0^{1,p}(\Omega) \times \mathcal{W}_0^{1,p}(\Omega)$ is a continuous operator, on the other hand $\mathcal{W}_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, hence the result. \square

Let $\mathcal{B}_1(u, v)(\cdot) := \tilde{\varphi}(\cdot, u(\cdot), v(\cdot))$ the Nemytskii operator associated to $\tilde{\varphi}$, and $\mathcal{B}_2(u, v)(\cdot) := \tilde{\psi}(\cdot, u(\cdot), v(\cdot))$ the Nemytskii operator associated to $\tilde{\psi}$.

Lemma 6 *The operators $\mathcal{B}_1 : L^p(\Omega) \times L^p(\Omega) \rightarrow L^r(\Omega)$, and $\mathcal{B}_2 : L^p(\Omega) \times L^p(\Omega) \rightarrow L^r(\Omega)$ are continuous and bounded.*

Proof. We see that $\frac{r}{p(p-1)} + \frac{p(p-1)-r}{p(p-1)} = 1$, using Young inequality in (\tilde{H}) with these exponents, we get

$$(\tilde{H}_1) \quad |\tilde{\varphi}(x, u, v)| \leq \frac{r}{p(p-1)} |u|^{\frac{p}{r}} + \frac{r}{p(p-1)} |v|^{\frac{p}{r}} + a(x)$$

where $a(x) = \frac{p(p-1)-r}{p(p-1)} a_1^{\frac{p(p-1)}{p(p-1)-r}} + \frac{p(p-1)-r}{p(p-1)} a_2^{\frac{p(p-1)}{p(p-1)-r}} + c(x)$. Since $r \leq p'$ and $|\Omega| < \infty$ then $L^{p'} \hookrightarrow L^r$, so $|h|^{p-1}, |k|^{p-1} \in L^r$ consequently $a \in L^r$. According to [26, proposition 26.6] an increase of type (\tilde{H}_1) is sufficient for the operator \mathcal{B}_1 to be continuous and bounded from $L^p(\Omega) \times L^p(\Omega)$ to

$L^r(\Omega)$, by the same way we show that \mathcal{B}_2 is continuous and bounded from $L^p(\Omega) \times L^p(\Omega)$ to $L^r(\Omega)$. \square

Proof of the theorem. Let $\Lambda(f, g) := (\mathcal{B}_1 \circ T(f, g); \mathcal{B}_2 \circ T(f, g))$

For $M > 0$, we put:

$$\mathcal{K}_M = \{(f, g) \in L^r(\Omega) \times L^r(\Omega) : \|(f, g)\|_{r \times r} \leq M\}$$

where $\|(f, g)\|_{r \times r} = \max\{\|f\|_r; \|g\|_r\}$.

For $(f, g) \in \mathcal{K}_M$, we have:

$$\begin{aligned} \|\Lambda(f, g)\|_{r \times r} &= \|(\mathcal{B}_1 \circ T(f, g); \mathcal{B}_2 \circ T(f, g))\|_{r \times r} \\ &= \max\{\|\mathcal{B}_1(\tilde{u}_f, \tilde{v}_g)\|_r; \|\mathcal{B}_2(\tilde{u}_f, \tilde{v}_g)\|_r\}. \end{aligned}$$

Using (\tilde{H}) we get:

$$\|\mathcal{B}_1(\tilde{u}_f, \tilde{v}_g)\|_r \leq a'_1 \|\tilde{u}_f\|_{r(p-1)}^{p-1} + a'_2 \|\tilde{v}_g\|_{r(p-1)}^{p-1} + \|c\|_r$$

according to ([8, theorem 2.5]) we know that there exist $C > 0$ such that

$$\|\tilde{u}_f\|_s^{p-1} \leq C \|f\|_r$$

and

$$\|\tilde{v}_g\|_s^{p-1} \leq C \|g\|_r$$

Since $s > r(p-1)$ and $|\Omega| < \infty$, the Hölder inequality implicate that

$$\begin{aligned} \|\tilde{u}_f\|_{r(p-1)}^{p-1} &\leq |\Omega|^{\frac{1}{r} - \frac{(p-1)}{s}} \|\tilde{u}_f\|_s^{p-1} \\ &\leq C |\Omega|^{\frac{p}{d}} \|f\|_r \end{aligned}$$

and

$$\begin{aligned} \|\tilde{v}_g\|_{r(p-1)}^{p-1} &\leq |\Omega|^{\frac{1}{r} - \frac{(p-1)}{s}} \|\tilde{v}_g\|_s^{p-1} \\ &\leq C |\Omega|^{\frac{p}{d}} \|g\|_r \end{aligned}$$

so

$$\|\mathcal{B}_1 \circ (f, g)\|_r \leq C |\Omega|^{\frac{p}{d}} (a'_1 \|f\|_r + a'_2 \|g\|_r) + \|c\|_r$$

and by the same way, we get

$$\|\mathcal{B}_2 \circ (f, g)\|_r \leq C |\Omega|^{\frac{p}{d}} \left(b'_1 \|f\|_r + b'_2 \|g\|_r \right) + \|c'\|_r$$

since $|\Omega|$ is small enough we can choose $|\Omega|$ such that:

$$\lambda := \max \left\{ a'_1, a'_2, b'_1, b'_2 \right\} C |\Omega|^{\frac{p}{d}} < 1$$

hence

$$\|\mathcal{B}_1 \circ (f, g)\|_r \leq \lambda \max \{ \|f\|_r, \|g\|_r \} + \|c\|_r$$

and

$$\|\mathcal{B}_2 \circ (f, g)\|_r \leq \lambda \max \{ \|f\|_r, \|g\|_r \} + \|c'\|_r$$

so

$$\max \{ \|\mathcal{B}_1 \circ (f, g)\|_r ; \|\mathcal{B}_2 \circ (f, g)\|_r \} \leq \lambda \max \{ \|f\|_r, \|g\|_r \} + \max \left\{ \|c\|_r ; \|c'\|_r \right\}$$

then

$$\begin{aligned} \|\Lambda(f, g)\|_{r \times r} &\leq \lambda \max \{ \|f\|_r, \|g\|_r \} + \max \left\{ \|c\|_r ; \|c'\|_r \right\} \\ &\leq \lambda M + \max \left\{ \|c\|_r ; \|c'\|_r \right\} \end{aligned}$$

putting

$$M_0 := \frac{\max \left\{ \|c\|_r ; \|c'\|_r \right\}}{1 - \lambda}$$

hence $\Lambda(\mathcal{K}_M) \subset \mathcal{K}_M$, $\forall M \geq M_0$.

\mathcal{K}_M is a non empty closed convex subset of $L^r(\Omega) \times L^r(\Omega)$ and $\Lambda : L^r(\Omega) \times L^r(\Omega) \rightarrow L^r(\Omega) \times L^r(\Omega)$ is completely continuous, so by Schauder fixed point theorem Λ admits a fixed point in \mathcal{K}_M , hence the problem (P) admits a weak solution. \square

By the regularity theory [20, Corollary 4.10], any bounded solution of (1) can be redefined in a set of measure zero so that it becomes continuous.

References

- [1] K. Adriouch, *On Quasilinear and Anisotropic Elliptic Systems with Sobolev Critical Exponents*, Ph.D. thesis, 2007.

- [2] A. Baalal, *Théorie du potentiel pour des opérateurs elliptiques non linéaires du second ordre à coefficients discontinus*, Potential Analysis 15: 255-271, 2001.
- [3] A. BAALAL & N. BelHaj Rhouma, *Dirichlet problem for quasi-linear elliptic equations*, *Electronic Journal of Differential Equations*, Vol. **2002** (2002), No. 82, pp. 1-18.
- [4] A. Bechah, K. Chaïb and F. de Thélin, *Existence and uniqueness of positive solution for subhomogeneous elliptic problems in \mathbb{R}^n* , Rev. Mat. Apl., 21 (1-2) (2000) 1-17.
- [5] L. Boccardo and D. G. de Figueiredo, *Some remarks on a system of quasilinear elliptic equations* NoDEA: Nonlin. Diff. Equ. and Appl., 2002 - Springer.
- [6] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York Dordrecht Heidelberg London, 2011.
- [7] J. Chabrowski, *On multiple solutions for nonhomogeneous system of elliptic equations*, Rev. Mat. Univ. Complut. Madrid, 9 (1), (1996) 207–234.
- [8] D. Daners & P. Drábek, *A priori estimates for a class of quasi-linear elliptic equations*, American Mathematical Society, Volume 361, Number 12, December 2009, Pages 6475-6500.
- [9] F. Demengel and G. Demengel, *Functional Spaces for the Theory of Elliptic Partial Differential Equations*, Springer-Verlag London Limited 2012.
- [10] F. de Thélin, *Quelques résultats d'existence et de nonexistence pour une EDP elliptique nonlinéaire*, C. R. Acad. Sci. Paris. 229 Série I, 18: 839–844, 1984.
- [11] F. de Thélin, *Résultats d'existence et de non-existence pour la solution positive et bornée d'une EDP elliptique non-linéaire*, Ann. Fac. Sc. Toulouse. 8 (3) (1987) 375–389.
- [12] F. de Thélin, *Première valeur propre d'un système elliptique non linéaire*, C.R. Acad. Sci. Paris Sér. I Math. 311 (1990), no. 10, 603–606.

- [13] F. de Thélin and J. Velin, *Existence and non-existence of nontrivial solutions for some nonlinear elliptic systems*, Rev. Matemática de la Universidad Complutense de Madrid 6 (1993), 153–154.
- [14] J. I. Díaz and J. E. Saa, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilineaires*, C. R. Acad. Sci. Paris, 305 Série I (1987) 521–524.
- [15] P. Drábek, *The p -Laplacian – Mascot of Nonlinear Analysis*, Acta Math. Univ. Comenianae, Vol. LXXVI, 1(2007), pp. 85–98.
- [16] P. L. Felmer, R. F. Manásevich and F. de Thélin, *Existence and uniqueness of positive solutions for certain quasilinear elliptic systems*, Comm. Part. Diff.Equ, 17 (11-12), (1992) 2013–2029.
- [17] P. L. Felmer, R. F. Manásevich and F. de Thélin, *Existence and uniqueness of positive solutions for certain quasilinear elliptic systems*, Comm. Part. Diff.Equ., 17 (11-12), (1992) 2013–2029.
- [18] D. Gilbart and N.S. Trudinger, *Elliptic Partial Differential Equations of Second order*, Springer-Verlag , Berlin Heidelberg New York, 2001.
- [19] J. Heinonen, T. Kilpelainen, O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Clarendon Press, Oxford New York Tokyo, 1993.
- [20] J. Malý and W. P. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Math. Surveys Monographs 51, Amer. Math. Soc., 1997.
- [21] Vladimir G. Maz'ja, *Sobolev Spaces*, Springer-Verlag Berlin Heidelberg, 1985.
- [22] V. Maz'ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, Springer-Verlag Berlin Heidelberg 1985, 2011.
- [23] M. Ôtani, *On certain second order ordinary differential equations associated with Sobolev-Poincaré-type inequalities*, Nonlinear Anal. 8 (11), (1984), 1255–1270.

- [24] M. Ôtani, *Existence and nonexistence of nontrivial solution of some nonlinear degenerate elliptic equations*, J. Func. Anal., 76 (1),(1988), 140–159 .
- [25] W. M. Ni J. Serrin, *Existence and nonexistence theorems for ground states of quasilinear partial differential equations of the anomalous case*, Acad. Naz.Lincei 77, 231–287, 1986.
- [26] E. Zeidler, *Nonlinear Functional Analysis and its Applications, II/B: Nonlinear Monotone Operators*, Springer-Verlag, 1990.